String order and symmetries in quantum spin lattices. PRL 100,167202 (2008)

FCS (finite correlated state)

Short correlation: non-degenerate maximum eigenvalue of modulus one. (injective, for a brief explanation, see prb 83,035107)

Positive: $\lambda \geq 0$, (left fix point vector, right is 1.)

Pure state: $\Lambda > 0$

$\mathcal{E}_u(X) = \sum_{n,n'} A_n X A_{n'}^\dagger \langle n'|u|n\rangle$
$= \sum_{j,j'} (\sum_j \langle j|n\rangle A_n) X (\sum_{j'} \langle n'|j'\rangle A_{n'}^\dagger) \langle j'|u|j\rangle$
$= \sum_j e^{i\theta_j} A_j X A_j^\dagger$

where, $\{|j\rangle\}$ are eigenvectors of unitary $u : u|j\rangle = e^{i\theta_j}|j\rangle$

Lemma 1:

1), $\rho(\mathcal{E}_u) \leq 1$;

2), If exist eigenvalue λ, $|\lambda| = 1$, then λ is the unique eigenvalue with modulus one, the correspond eigenvector V , which is unitary, is not degenerate. This statement is equivenlent to the condition: $V^\dagger A_j = e^{i(\theta-\theta_j)} A_j V^\dagger$

Proof:

Let, $\mathcal{E}_u(V) = \lambda V$,

Consider, $tr(V\Lambda V^\dagger) = tr(V^\dagger V \Lambda) > 0$, since, $\Lambda > 0, V^\dagger V \geq 0$ (positive)

$|\lambda| tr(V\Lambda V^\dagger) = |\sum_j e^{i\theta_j} tr(A_j V A_j^\dagger \Lambda V^\dagger)|$
$= |\sum_j tr[(V A_j^\dagger \sqrt{\Lambda})(e^{i\theta_j} \sqrt{\Lambda} V^\dagger A_j)]|$
$\equiv |\sum_j tr(\alpha_j^\dagger \beta_j)|$

Where, $\alpha_j = \sqrt{\Lambda} A_j V^\dagger$, $\beta_j = e^{i\theta_j} \sqrt{\Lambda} V^\dagger A_j$,

$|\sum_j tr(\alpha_j^\dagger \beta_j)| \leq \sqrt{\sum_j tr(\alpha_j^\dagger \alpha_j)} \cdot \sqrt{\sum_j tr(\beta_j^\dagger \beta_j)}$,

i.e,

$|\lambda| tr(V\Lambda V^\dagger) \leq \sqrt{\sum_j tr(V A_j^\dagger \Lambda A_j V^\dagger)} \cdot \sqrt{\sum_j tr(A_j^\dagger V \Lambda V^\dagger A_j)} = tr(V\Lambda V^\dagger)$

Thus, $|\lambda| \leq 1$, i.e, $\rho(\mathcal{E}_u) \leq 1$

(Note, we have inner product $\langle \alpha|\beta\rangle = \sum_j tr(\alpha_j^\dagger \beta_j)$.)

When, $|\lambda| = 1$, it means α, β linear dependent, i.e,

$\beta = k\alpha$, i.e, $\beta_j = k\alpha_j, \forall j$,

Thus,

$kA_jV^\dagger = e^{i\theta_j}V^\dagger A_j$,

$\bar{k}\sum_j e^{i\theta_j}A_jVA_j^\dagger = \sum_j A_jA_j^\dagger V$,

$\bar{k}\lambda V = V$,

Thus, $|k| = 1$,

Let, $k = e^{i\theta}$,

We have,

$V^\dagger A_j = e^{i(\theta-\theta_j)}A_jV^\dagger$

Moreover, consider,

$\mathcal{E}(V^\dagger V) = \sum_j A_jV^\dagger VA_j^\dagger = \sum_j e^{-i(\theta-\theta_j)}V^\dagger A_je^{i(\theta-\theta_j)}A_j^\dagger V = V^\dagger\sum_j A_jA_j^\dagger V = V^\dagger V$

Since $\mathcal{E}$ has unique eigenvalue of modulus one (pure state), i.e, $\mathcal{E}(1) = 1$

$V^\dagger V = 1$

i.e, V is unitary.

On the contary, if, we have,

$V^\dagger A_j = e^{i(\theta-\theta_j)}A_jV^\dagger$

First, follow the argument above, V is unitary.

Then,

$e^{i\theta}A_j^\dagger V = e^{i\theta_j}VA_j^\dagger$

$e^{i\theta}(\sum_j A_jA_j^\dagger)V = \sum_j e^{i\theta_j}A_jVA_j^\dagger$

$e^{i\theta}V = \mathcal{E}_u(V)$

Thus, V is the eigenvector of $\mathcal{E}_u$ with eigenvalue $\lambda = e^{i\theta}$, i.e, $|\lambda| = 1$

Finally, we prove, for $\mathcal{E}_u$, Eigenvalue of modulus one is unique.

Suppose we have unitary, V, V' and eigenvalue $e^{i\theta}, e^{i\theta'}$

Since, $|e^{i\theta}| = |e^{i\theta'}| = 1$, must satisfy,

$V^\dagger A_j = e^{i(\theta-\theta_j)}A_jV^\dagger$

$V'^\dagger A_j = e^{i(\theta'-\theta_j)}A_jV'^\dagger$

$\mathcal{E}(V^\dagger V') = \sum_j A_j V^\dagger V' A_j^\dagger = e^{i(\theta'-\theta)} V^\dagger V'$

Since $\mathcal{E}$ has unique eigenvalue of modulus one, i.e, $\lambda_{max} = 1$,

$\theta = \theta', V^\dagger V' = 1$

Consider, V, V' are unitary,

$V = V'$

QED.

Additionally,

$\mathcal{E}_u^*(X) = \sum_{n,n'} A_n^\dagger X A_{n'} \langle n|u|n'\rangle$
$= \sum_{j,j'} (\sum_{j'} \langle n|j'\rangle A_n^\dagger) X (\sum_j \langle j|n'\rangle A_{n'}) \langle j'|u|j\rangle$
$= \sum_j e^{i\theta_j} A_j^\dagger X A_j$

If $\rho(\mathcal{E}_u) = 1$, i.e, exist unique $\lambda = e^{i\theta}, \leftrightarrow V^\dagger A_j = e^{i(\theta-\theta_j)} A_j V^\dagger$, we also have,

$\mathcal{E}_u^*(\Lambda V^\dagger) = e^{i\theta} \Lambda V^\dagger$

Thus, $\langle \Psi_L|u^{\otimes L}|\Psi_L\rangle = e^{iL\theta} tr(\Lambda V^\dagger V) = e^{iL\theta}$

Else, if $\rho(\mathcal{E}_u) < 1$, $lim_{L\to\infty} \langle \Psi_L|u^{\otimes L}|\Psi_L\rangle = 0$

$tr[\Lambda V^\dagger \mathcal{E}_y(1)]^*$
$= \sum_{nn'} tr[\Lambda V^\dagger \langle n'|y|n\rangle A_n A_{n'}^\dagger]^*$
$= \sum_{nn'} \langle n|y^\dagger|n'\rangle tr[\Lambda V^\dagger A_n A_{n'}^\dagger]^*$

use, $V^\dagger A_n = e^{i(\theta-\theta_n)} A_n V^\dagger$

$= \sum_{nn'} \langle n|y^\dagger|n'\rangle e^{i(\theta_n-\theta)} tr[\Lambda A_n V^\dagger A_{n'}^\dagger]^*$
$= \sum_{nn'} \langle n|y^\dagger|n'\rangle e^{i(\theta_n-\theta)} tr[A_{n'}^* V^* A_n^T \Lambda]^*$
$= \sum_{nn'} \langle n|e^{-i\theta} u y^\dagger|n'\rangle tr[A_{n'} V A_n^\dagger \Lambda]$
$= tr[\Lambda \mathcal{E}_z(V)]$

Where, $z \equiv e^{-i\theta} u y^\dagger \equiv \tilde{u} y^\dagger$

we can let $x = \tilde{u} y^\dagger$, i.e, $y = x^\dagger \tilde{u}$

Thus,

$tr[\Lambda \mathcal{E}_x(V)] = tr[\Lambda V^\dagger \mathcal{E}_y(1)]^*$

consider $x = |n\rangle\langle m|$

$tr[\Lambda \mathcal{E}_x(V)] = tr[\Lambda A_m V A_n^\dagger]$

Since, $V^\dagger A_n V = e^{i(\theta-\theta_n)} A_n \to A_m V = e^{i(\theta-\theta_m)} V A_m$

$tr[\Lambda\mathcal{E}_x(V)] = e^{i(\theta-\theta_m)} tr[\Lambda V A_m A_n^\dagger]$

Theorem 1:

For a pure FCS,

Exists SO iff exist unitary $\tilde{u} \neq 1$, and n,m, satisfies, 1), $\mathcal{E}_{\tilde{u}}(V) = V$, i.e, $\rho(\mathcal{E}_u) = 1$; 2), $tr[\Lambda V A_m A_n^\dagger] \neq 0$, i.e, $x = |n\rangle\langle m|, y = x^\dagger \tilde{u}$

($\tilde{u}$ is diagonal under basis $\{|n\rangle\}$)

FCS has local symmetry: $\exists u, u^{\otimes N}|\Psi\rangle = e^{i\phi}|\Psi\rangle, i.e, u^{\otimes N}\rho u^{\dagger\otimes N}$

If $\rho(\mathcal{E}_u) = 1$, then, $\langle\Psi|u^{\otimes N}|\Psi\rangle = e^{iN\theta}$, thus, has local symmetry, else, $\langle\Psi|u^{\otimes N}|\Psi\rangle \to 0$

Thus, we have theorem,

Theorem 2:

A pure FCS has a local symmetry iff $\rho(\mathcal{E}_u) = 1$ (i.e, $A_j = e^{i(\theta-\theta_j)} V A_j V^\dagger$, i.e, C1) condition.)

We can show the graph.

On the contrary, if FCS has local symmetry,
$0 \neq 1 = tr(\rho) = tr(\rho^2) = tr(\rho u^{\otimes N}\rho u^{\dagger\otimes N}) = \langle\Psi|u^{\otimes N}|\Psi\rangle\langle\Psi|u^{\dagger\otimes N}|\Psi\rangle = tr[\Lambda\mathcal{E}_u{}^{\otimes N}(1)] tr[\Lambda\mathcal{E}_{u^\dagger}{}^{\otimes N}(1)]$

Thus, $\rho(\mathcal{E}_u) = 1$

Define,

isometry: $B = \sum_j |j\rangle A_j$

$E = \sum_j A_j \otimes \bar{A}_j$

if FCS has local symmetry, consider u is an element of of a unitary representation of the symmetry group, according to lemma 2, we have unique V, we have the following three conditions.

C1). $(u \otimes 1)B = e^{i\theta}(1 \otimes V)BV^\dagger$

C2). $\mathcal{E}(VXV^\dagger) = V\mathcal{E}(X)V^\dagger$

C3). $[E, (V \otimes \bar{V})] = 0$

Proof:

1. First, we can always consider the basis where u is diagonal, since, $B = \sum_j |j\rangle A_j = \sum_n |n\rangle(\sum_j \langle n|j\rangle A_j) = \sum_n |n\rangle A_n$

$(u\otimes 1)B=\sum_j u|j\rangle A_j=\sum_j |j\rangle e^{i\theta_j}e^{i(\theta-\theta_j)}VA_jV^\dagger=e^{i\theta}(1\otimes V)BV^\dagger$

2. $\mathcal{E}(VXV^\dagger)$
$=\sum_j A_jVXV^\dagger A_j^\dagger$
$=\sum_j e^{i(\theta-\theta_j)}VA_jV^\dagger VXV^\dagger e^{-i(\theta-\theta_j)}VA_j^\dagger V^\dagger$
$=V\sum_j A_jXA_j^\dagger V^\dagger=V\mathcal{E}(X)V^\dagger$
3. $[E,(V\otimes\bar{V})]$
$=EV\otimes\bar{V}-V\otimes\bar{V}E$
$=\sum_j A_jV\otimes\bar{A}_j\bar{V}-\sum_j VA_j\otimes\bar{V}\bar{A}_j$
$=\sum_j e^{i(\theta-\theta_j)}VA_j\otimes e^{-i(\theta-\theta_j)}\bar{V}\bar{A}_j-\sum_j VA_j\otimes\bar{V}\bar{A}_j$
$=0$

Note,

C1) means that V is a (projective) unitary representation, since,
$(u'u\otimes 1)B=e^{i(\theta+\theta')}(1\otimes V'V)B(V'V)^\dagger$

C2) means that $\mathcal{E}$ is a group isomorphism, $X\to\mathcal{E}(X)$

C3) means that E is a Carsimir operator.

Consider the continuous symmetry, $V=e^{i\phi H}$

$[E,(1+i\phi H)\otimes(1-i\phi\bar{H})]=i\phi[E,H\otimes 1-1\otimes H]=0$

Define, liner map, $M(H)=[E,H\otimes 1-1\otimes H]$

Thus, FCS has continuous symmetry iff the map M has a nontrivial kernel, i.e,
$\exists H\neq 0, M(H)=0$

Note, for continuous symmetry, we can always have SO, since we can choose ϕ small enough, such that, $tr[V\Lambda]=tr[\Lambda]+i\phi tr[\Lambda H]\neq 0$